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# SERIES RELATIONS COMING FROM CERTAIN FUNCTIONS RELATED TO GENERALIZED NON-HOLOMORPHIC EISENSTEIN SERIES

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Abstract. Using a modular transformation formula for a class of functions related to generalized non-holomorphic Eisenstein series, we find a new class of infinite series about identities, some of which include generalized formulae of several Berndt's results.

## 1. Introduction

In [4], the author proved a modular transformation formula for a class of functions which stem from generalized non-holomorphic Eisenstein series. Using this formula, we found a few class of new infinite series identities [5, 6]. In fact, these works were motivated by the works of B. C. Berndt who established many relations between various infinite series [2, 3]. He used a transformation formula for a large class of functions that comes from a more general class of Eisenstein series. He says the flavor of his results is much like those infinite series identities found in Ramanujan's Notebooks [7]. We hereby state three identities of Berndt's results in [3], of which generalized formulae will be given in this paper. For $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$,

$$\alpha \sum_{m=0}^{\infty} \operatorname{sech}^2\left(\frac{1}{2}(2m+1)\alpha\right) + \beta \sum_{m=0}^{\infty} \operatorname{sech}^2\left(\frac{1}{2}(2m+1)\beta\right) = 1. \tag{1.1}$$

$$\sum_{m=0}^{\infty} \operatorname{sech}^2\left(\frac{1}{2}(2m+1)\pi\right) = \frac{1}{2\pi}. \tag{1.2}$$

$$\alpha \sum_{m=1}^{\infty} \operatorname{sech}^2(m\alpha) - \beta \sum_{m=0}^{\infty} \operatorname{csch}^2\left(\frac{1}{2}(2m+1)\beta\right) = 1 - \frac{\alpha}{2}. \tag{1.3}$$

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He obtained (1.1) and (1.3) by differentiation of the transformation formulae for the theta-functions. We see that if $\alpha = \beta = \pi$ in (1.1), then (1.2) follows. Berndt [3] says that several other authors have proved (1.2) by their own methods earlier than his work. In this paper, we continue the study made in [5, 6]. We derive a few class of infinite series identities. Some of these results give elegant types of generalization of (1.1), (1.2) and (1.3). In particular, it is interesting to see that our generalized formulae of (1.1) and (1.2) look quite different(Corollary 3.3, Corollary 3.4) in consideration of the fact that (1.2) easily comes from (1.1). We derive these identities directly from only one source, our transformation formula without differentiation of the transformation formula. It is also noteworthy that most of our results appear to be new.

## 2. Notation

In this section, we introduce the necessary notation and shall state the principal theorem which we use to obtain our results. Let $\mathbb{Z}$ and $\mathbb{C}$ be the set of integers and the set of complex numbers, respectively. For $z \in \mathbb{C}$, we choose the branch of the argument defined by $-\pi \leq \arg z < \pi$. Let $\Gamma(s)$ denote the Gamma function. For any non-negative integer $n$, the rising factorial $(x)_n$ is defined by

$$(x)_n = x(x+1)\cdots(x+n-1) \text{ for } n > 0, \ (x)_0 = 1.$$

It is easy to see that

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}. \tag{2.1}$$

The confluent hypergeometric function of the first kind ${}_1F_1(\alpha;\beta;z)$ is defined by

$${}_1F_1(\alpha;\beta;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n n!} z^n$$

and the confluent hypergeometric function of the second kind $U(\alpha,\beta,z)$ is defined to be

$$U(\alpha,\beta,z) = \frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\ {}_1F_1(\alpha;\beta;z) + \frac{\Gamma(\beta-1)}{\Gamma(\alpha)} z^{1-\beta}\ {}_1F_1(1+\alpha-\beta;2-\beta;z).$$

The function $U(\alpha,\beta,z)$ can be analytically continued to all values of $\alpha$, $\beta$ and $z$ real or complex [8]. Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, the upper half-plane. For real $r_k$ and $h_k (k = 1, 2)$, let $r = (r_1, r_2)$ and $h = (h_1, h_2)$. Let $e(x) = e^{2\pi i x}$. For $\tau \in \mathbb{H}$ and $s_1,\ s_2 \in \mathbb{C}$ with $s = s_1 + s_2$, define

$$\mathcal{A}(\tau, s_1, s_2; r, h) = \sum_{m+r_1>0} \sum_{n-h_2>0} \frac{e\left(mh_1 + ((m + r_1)\tau + r_2)(n - h_2)\right)}{(n - h_2)^{1-s}} \times U(s_2; s; 4\pi(m + r_1)(n - h_2)\mathrm{Im}(\tau))$$

and

$$\bar{\mathcal{A}}(\tau, s_1, s_2; r, h) = \sum_{m+r_1>0} \sum_{n+h_2>0} \frac{e\left(mh_1 - ((m + r_1)\bar{\tau} + r_2)(n + h_2)\right)}{(n + h_2)^{1-s}} \times U(s_1; s; 4\pi(m + r_1)(n + h_2)\mathrm{Im}(\tau)).$$

Let

$$\mathcal{H}(\tau, s_1, s_2; r, h) = \mathcal{A}(\tau, s_1, s_2; r, h) + e^{\pi i s}\mathcal{A}(\tau, s_1, s_2; -r, -h)$$

and

$$\bar{\mathcal{H}}(\tau, s_1, s_2; r, h) = \bar{\mathcal{A}}(\tau, s_1, s_2; r, h) + e^{\pi i s}\bar{\mathcal{A}}(\tau, s_1, s_2; -r, -h).$$

Let

$$\mathbf{H}(\tau, \bar{\tau}, s_1, s_2; r, h) = \frac{1}{\Gamma(s_1)}\mathcal{H}(\tau, s_1, s_2; r, h) + \frac{1}{\Gamma(s_2)}\bar{\mathcal{H}}(\tau, s_1, s_2; r, h).$$

For real $x$, $\alpha$ and $t \in \mathbb{C}$ with Re $t > 1$, let

$$\psi(x, \alpha, t) = \sum_{n+\alpha>0} \frac{e(nx)}{(n + \alpha)^t}$$

and

$$\Psi(x, \alpha, t) = \psi(x, \alpha, t) + e^{\pi i t}\psi(-x, -\alpha, t),$$
$$\Psi_{-1}(x, \alpha, t) = \psi(x, \alpha, t - 1) + e^{\pi i t}\psi(-x, -\alpha, t - 1).$$

The characteristic function of the integers is defined by $\lambda$. For a real number $x$, $[x]$ denotes the greatest integer less than or equal to $x$ and $\{x\} = x - [x]$. Let

$$V\tau = \frac{a\tau + b}{c\tau + d}$$

denote a modular transformation with $c > 0$ for $\tau \in \mathbb{C}$. Let

$$R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

and

$$H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

We now can state the principal theorem.

THEOREM 2.1. [5]. *Let $Q = \{\tau \in \mathbb{H} \mid \text{Re } \tau > -d/c\}$ and $\varrho = c\{R_2\} - d\{R_1\}$. Let $s_1, s_2 \in \mathbb{C}$ with $s = s_1 + s_2$ and assume that $s$ is not an integer less than or equal to 1. Then, for $\tau \in Q$,*

$$\begin{aligned} z^{-s_1}\bar{z}^{-s_2}\mathbf{H}(V\tau, V\bar{\tau}, s_1, s_2; r, h) &= \mathbf{H}(\tau, \bar{\tau}, s_1, s_2; R, H) \\ &+\lambda(R_1)e(-R_1H_1)(2\pi i)^{-s}e^{-\pi i s_2}\Psi(-H_2, -R_2, s) \\ &-\lambda(r_1)e(-r_1h_1)(2\pi i)^{-s}e^{\pi i s_1}z^{-s_1}\bar{z}^{-s_2}\Psi(h_2, r_2, s) \\ &+\lambda(H_2)(4\pi\text{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)}\Psi_{-1}(H_1, R_1, s) \\ &-\lambda(h_2)(4\pi\text{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)}z^{s_2-1}\bar{z}^{s_1-1}\Psi_{-1}(h_1, r_1, s) \\ &+\frac{(2\pi i)^{-s}e^{-\pi i s_2}}{\Gamma(s_1)\Gamma(s_2)}\mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H), \end{aligned}$$

*where $z = c\tau + d$ and*

$$\begin{aligned} &\mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H) \\ &= \sum_{j=1}^{c} e(-H_1(j + [R_1] - c) - H_2([R_2] + 1 + [(jd + \varrho)/c] - d)) \\ &\quad \times \int_0^1 v^{s_1-1}(1-v)^{s_2-1}\int_C u^{s-1}\frac{e^{-(zv+\bar{z}(1-v))(j-\{R_1\})u/c}}{e^{-(zv+\bar{z}(1-v))u} - e(cH_1 + dH_2)} \\ &\qquad\qquad \times \frac{e^{\{(jd+\varrho)/c\}u}}{e^u - e(-H_2)}\, dudv, \end{aligned}$$

*where $C$ is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u = 0$ is the only zero of*

$$(e^{-(zv+\bar{z}(1-v))u} - e(cH_1 + dH_2))(e^u - e(-H_2))$$

*lying inside the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of $u^s$ with $0 < \arg u < 2\pi$.*

Let $B_n(x)$ denote the $n$-th Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \ (|t| < 2\pi).$$

The $n$-th Bernoulli number $B_n$, $n \geq 0$, is defined by $B_n = B_n(0)$. Put $\bar{B}_n(x) = B_n(\{x\})$, $n \geq 0$. Let ${}_2F_1(\alpha, \beta; \gamma; z)$ be a hypergeometric function defined by

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}z^n,$$

which has the following integral representation. For $\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0$ and $z \in \mathbb{C}\backslash[1,\infty)$([1], p. 65),

$$ {}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta}\, dt. $$

REMARK 2.2. Let $s = s_1 + s_2$ be an integer. Suppose $H_1 = H_2 = 0$. By the residue theorem, we find that

$$ \begin{aligned} &\int_C u^{s-1} \frac{e^{-(zv+\bar{z}(1-v))(j-\{R_1\})u/c}}{e^{-(zv+\bar{z}(1-v))u}-1} \frac{e^{\{(\varrho+jd)/c\}u}}{e^u-1}\, dudv \\ &= 2\pi i \sum_{k=0}^{-s+2} \frac{B_k((j-\{R_1\})/c)\bar{B}_{-s+2-k}((\varrho+jd)/c)}{k!(-s+2-k)!}(-z)^{k-1}. \end{aligned} $$

We see that if $s > 2$, then the above sum containing Bernoulli polynomials vanishes and so $\frac{1}{\Gamma(s_1)(s_2)}\mathbf{L}(\tau,\bar{\tau},s_1,s_2;R,H)$ can be defined for all $\tau \in \mathbb{H}$.

Furthermore, we have

$$ \begin{aligned} &\int_0^1 v^{s_1-1}(1-v)^{s_2-1} \int_C u^{s-1} \frac{e^{-(zv+\bar{z}(1-v))(j-\{R_1\})u/c}}{e^{-(zv+\bar{z}(1-v))u}-1} \frac{e^{\{(\varrho+jd)/c\}u}}{e^u-1}\, dudv \\ &= 2\pi i \sum_{k=0}^{-s+2} \frac{B_k((j-\{R_1\})/c)\bar{B}_{-s+2-k}((\varrho+jd)/c)}{k!(-s+2-k)!}(-z)^{k-1} \\ &\qquad \times \int_0^1 (1-v)^{s_1-1}v^{s_2-1}\left(1-\frac{z-\bar{z}}{z}v\right)^{k-1} dv \\ &= 2\pi i \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s)} \sum_{k=0}^{-s+2} \frac{B_k((j-\{R_1\})/c)\bar{B}_{-s+2-k}((\varrho+jd)/c)}{k!(-s+2-k)!} \\ &\qquad \times(-z)^{k-1}\, {}_2F_1\left(s_2, 1-k; s; \frac{z-\bar{z}}{z}\right). \end{aligned} $$

Thus we see that $\frac{1}{\Gamma(s_1)(s_2)}\mathbf{L}(\tau,\bar{\tau},s_1,s_2;R,H)$ can be defined for all values of $s_1$ and $s_2$ with $s = s_1 + s_2 \in \mathbb{Z}$ and $\frac{c(\tau-\bar{\tau})}{c\tau+d} \in \mathbb{C}\backslash[1,\infty)$. If $s_2$ is an non-positive integer and $s > 0$, then it can be defined for all $\tau \in \mathbb{H}$.

## 3. A class of infinite series identities

In this section, we obtain a class of new infinite series identities using Theorem 2.1. The Eulerian number $E(n,j)$ is defined to be the number of permutations of numbers from 1 to $n$ such that exactly $j$ numbers are greater than the previous elements. Note that $E(n,j)=E(n,n-j-1)$. For any integer $n$, the polylogarithm function $\mathrm{Li}_n(z)$ is defined by

$$\mathrm{Li}_n(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^n},$$

where $z\in\mathbb{C}$ and $|z|<1$. For $n>0$, we see that

$$\mathrm{Li}_{-n}(z)=\frac{1}{(1-z)^{n+1}}\sum_{j=0}^{n-1}E(n,j)z^{n-j}.$$

From now on, we set

$$V\tau=\frac{\tau-1}{c\tau-c+1},$$

where $c$ is an positive integer. For any integer $k\geq 1$, let

$$\mu_k=\begin{cases}\frac{1}{2}, & k \text{ odd},\\ 0, & \text{even}.\end{cases}$$

THEOREM 3.1. *Let $\alpha,\ \beta>0$ with $\alpha\beta=\pi^2$. For any integers $B\geq 0$ and $N\geq 1$,*

$$\begin{aligned}(-1)^B\alpha^N\sum_{k=0}^{B}\binom{B}{k}\frac{(-\alpha/c)^k}{(2N+k-1)!}&\sum_{j=0}^{N+[k/2]-1}{}' E(2N+k-1,N+[k/2]-j-1)\\ &\times\sum_{m=0}^{\infty}\frac{\cosh((2m+1)(j+\mu_k)(\alpha-i\pi)/c)}{(2m+1)^{-k}\sinh^{2N+k}((2m+1)(\alpha-i\pi)/(2c))}\\ =(-\beta)^N\sum_{k=0}^{B}\binom{B}{k}\frac{(-\beta/c)^k}{(2N+k-1)!}&\sum_{j=0}^{N+[k/2]-1}{}' E(2N+k-1,N+[k/2]-j-1)\\ \times\sum_{m=0}^{\infty}&\frac{\cosh((2m+1)(j+\mu_k)(\beta+i\pi)/c)}{(2m+1)^{-k}\sinh^{2N+k}((2m+1)(\beta+i\pi)/(2c))}-\delta_N(B,c),\end{aligned}$$

*where $'$ means that if $k$ is even, then the term with $j=0$ is multiplied by $\frac{1}{2}$ and*

$$\delta_N(B,c)=\begin{cases}\frac{c}{4(B+1)}(1+(-1)^B), & N=1,\\ 0, & N\geq 2.\end{cases}$$

*Proof.* For $A,B,N\in\mathbb{Z}$, let $s_1=A\geq 1,\ s_2=-B\leq 0$ and $s=2N\geq 2$. Put $(r_1,r_2)=\left(\frac{1}{2},0\right),\ (h_1,h_2)=(0,0),\ c\tau-c+1=\frac{\pi}{\alpha}i$ in Theorem

2.1. Employing $V\tau = \frac{1}{c} + i\frac{\alpha}{c\pi}$, we see that

$$\mathcal{A}(V\tau, A, -B; r, h) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{e^{-(2m+1)n(\alpha-i\pi)/c}}{n^{1-2N}} U(-B; 2N; 2(2m+1)n\alpha/c).$$

Note that $\frac{1}{\Gamma(s_2)} = \frac{1}{\Gamma(-B)} = 0$. A direct calculation with (2.1) shows that

$$U(-B; 2N; 2(2m+1)n\alpha/c) = (-1)^B (A-1)! \sum_{k=0}^{B} \binom{B}{k} \frac{(-2(2m+1)n\alpha/c)^k}{(2N+k-1)!}.$$

Thus we have

$$\begin{aligned} \mathcal{A}(V\tau, A, -B; r, h) &= (-1)^B (A-1)! \sum_{k=0}^{B} \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!} \\ &\times \sum_{m=0}^{\infty} (2m+1)^k \mathrm{Li}_{-(2N+k-1)}(e^{-(2m+1)(\alpha-i\pi)/c}). \end{aligned} \tag{3.1}$$

Apply $E(2N+k-1, j) = E(2N+k-1, 2N+k-j-2)$ to obtain that

$$\begin{aligned} \mathrm{Li}_{-(2N+k-1)}(e^{-(2m+1)(\alpha-i\pi)/c}) &= \frac{2^{1-2N-k}}{\sinh^{2N+k}((2m+1)(\alpha-i\pi)/c)} \\ &\times \sum_{j=0}^{N+[k/2]-1}{}' E(2N+k-1, N+[k/2]-j-1) \\ &\times \cosh((2m+1)(j+\mu_k)(\alpha-i\pi)/c)). \end{aligned}$$

Using the above equation in (3.1), we find that

$$\begin{aligned} \mathcal{A}(V\tau, A, -B; r, h) &= \frac{(-1)^B (A-1)!}{2^{2N-1}} \sum_{k=0}^{B} \binom{B}{k} \frac{(-\alpha/c)^k}{(2N+k-1)!} \\ &\times \sum_{j=0}^{N+[k/2]-1}{}' E(2N+k-1, N+[k/2]-j-1) \\ &\times \frac{\cosh((2m+1)(j+\mu_k)(\alpha-i\pi)/c))}{(2m+1)^{-k}\sinh^{2N+k}((2m+1)(\alpha-i\pi)/c)}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} \mathcal{A}(V\tau, A, -B; -r, h) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{e^{-(2m+1)n(\alpha-i\pi)/c}}{n^{1-2N}} \\ &\qquad \times U(-B; 2N; 2(2m+1)n\alpha/c) \\ &= \mathcal{A}(V\tau, A, -B; r, h). \end{aligned}$$

Now recalling $\frac{1}{\Gamma(s_2)}=\frac{1}{\Gamma(-B)}=0$, we obtain that

$$\begin{aligned}\mathbf{H}(V\tau,V\bar{\tau};A,-B;r,h) &= \frac{2}{(A-1)!}\mathcal{A}(V\tau,A,-B;r,h)\\ &= \frac{(-1)^B}{2^{2N-2}}\sum_{k=0}^{B}\binom{B}{k}\frac{(-\alpha/c)^k}{(2N+k-1)!}\\ &\times\sideset{}{'}\sum_{j=0}^{N+[k/2]-1} E(2N+k-1,N+[k/2]-j-1)\\ &\times\frac{\cosh((2m+1)(j+\mu_k)(\alpha-i\pi)/c))}{(2m+1)^{-k}\sinh^{2N+k}((2m+1)(\alpha-i\pi)/c)}.\end{aligned}$$

Note that $(R_1,R_2)=(\frac{1}{2},-\frac{1}{2}),\ H=(0,0)$ and $\tau=1-\frac{1}{c}+i\frac{\pi}{c\alpha}$. By the same way to calculate $\mathcal{A}(V\tau,A,-B;r,h)$, we find

$$\begin{aligned}\mathcal{A}(\tau,A,-B;R,H) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{e^{-(2m+1)n(\beta-i\pi)/c}}{n^{1-2N}}U(-B;2N;2(2m+1)n\beta/c)\\ &= \frac{(-1)^B(A-1)!}{2^{2N-1}}\sum_{k=0}^{B}\binom{B}{k}\frac{(-\beta/c)^k}{(2N+k-1)!}\\ &\times\sideset{}{'}\sum_{j=0}^{N+[k/2]-1} E(2N+k-1,N+[k/2]-j-1)\\ &\times\frac{\cosh((2m+1)(j+\mu_k)(\beta+i\pi)/c))}{(2m+1)^{-k}\sinh^{2N+k}((2m+1)(\beta+i\pi)/c)}.\end{aligned}$$

We also see that $\mathcal{A}(\tau,A,-B;R,H)=\mathcal{A}(\tau,A,-B;-R,-H)$. Thus

$$\begin{aligned}\mathbf{H}(\tau,\bar{\tau};A,-B;R,H) &= \frac{2}{(A-1)!}\mathcal{A}(\tau,A,-B;R,H)\\ &= \frac{(-1)^B}{2^{2N-2}}\sum_{k=0}^{B}\binom{B}{k}\frac{(-\beta/c)^k}{(2N+k-1)!}\\ &\times\sideset{}{'}\sum_{j=0}^{N+[k/2]-1} E(2N+k-1,N+[k/2]-j-1)\\ &\times\frac{\cosh((2m+1)(j+\mu_k)(\beta+i\pi)/c))}{(2m+1)^{-k}\sinh^{2N+k}((2m+1)(\beta+i\pi)/c)}.\end{aligned}$$

The other parts of Theorem 2.1 are calculated as follows. It is easy to see that

$$(c\tau+d)^{-s_1}(c\bar{\tau}+d)^{-s_2}=\left(\frac{\pi}{\alpha}i\right)^{-A}\left(-\frac{\pi}{\alpha}i\right)^{B}=(-1)^B\alpha^N(-\beta)^{-N}.$$

Since $\lambda(R_1) = \lambda(\frac{1}{2}) = 0$ and $\lambda(r_1) = \lambda(\frac{1}{2}) = 0$, the equations with $\lambda(R_1)$ and $\lambda(r_1)$ vanish. Applying $\frac{1}{\Gamma(s_2)} = \frac{1}{\Gamma(-B)} = 0$, we also see that the equations with $\lambda(H_2)$ and $\lambda(h_2)$ are equal to 0. Using Remark 2.2, we find

$$\frac{(2\pi i)^{-s} e^{-\pi i s_2}}{\Gamma(s_1)\Gamma(s_2)} \mathbf{L}(\tau, \bar{\tau}, s_1, s_2, ; R, H)$$
$$= \frac{(-1)^B (2\pi i)^{1-2N}}{\Gamma(2N)} \sum_{\ell=1}^{c} \sum_{k=0}^{-2N+2} \frac{B_k((\ell-1/2)/c)\bar{B}_{-2N+2-k}((\ell-1/2)/c)}{k!(-2N+2-k)!}$$
$$\times \left(-\frac{\pi}{\alpha} i\right)^{k-1} {}_2F_1(-B, 1-k; 2N; 2).$$

Note that, in the above equation, the sum for the Bernoulli polynomials is valid only for $N = 1$. A short calculation shows that

$$ {}_2F_1(-B, 1; 2; 2) = \sum_{n=0}^{B} \binom{B}{n} \frac{(-2)^n}{n+1} = \frac{(-1)^B + 1}{2(B+1)}.$$

Finally, combining all the above results, we complete the proof. □

COROLLARY 3.2. *Let $\alpha$, $\beta > 0$ with $\alpha\beta = \pi^2$. For any integer $N \geq 1$,*

$$\alpha^{2N} \sum_{j=0}^{2N-1}{}' (-1)^j E(4N-1, 2N-1-j) \sum_{m=0}^{\infty} \frac{\cosh((2m+1)j\alpha)}{\cosh^{4N}((2m+1)\alpha/2)}$$
$$= \beta^{2N} \sum_{j=0}^{2N-1}{}' (-1)^j E(4N-1, 2N-1-j) \sum_{m=0}^{\infty} \frac{\cosh((2m+1)j\beta)}{\cosh^{4N}((2m+1)\beta/2)},$$

*where $'$ means that the term with $j = 0$ is multiplied by $\frac{1}{2}$.*

*Proof.* Let $B = 0$ and $c = 1$ in Theorem 3.1. We have

$$\cosh(((2m+1)j(\alpha \pm i\pi)) = (-1)^j \cosh((2m+1)j\alpha)$$

and

$$\sinh^{2N}\left(\frac{1}{2}(2m+1)(\alpha \pm i\pi)\right) = (-1)^N \cosh^{2N}\left(\frac{1}{2}(2m+1)\alpha\right).$$

Replace $N$ by $2N$ to obtain the desired result. □

Corollary 3.2 shows an elegant symmetric identity for $\alpha$ and $\beta$. Let $N = 1$ in Corollary 3.2. Using

$$\cosh x = 2\cosh^2\left(\frac{x}{2}\right) - \frac{1}{2},$$

we find

$$\alpha^2 \sum_{m=0}^{\infty} \operatorname{sech}^2\left(\frac{1}{2}(2m+1)\alpha\right) - \frac{5}{4}\alpha^2 \sum_{m=0}^{\infty} \operatorname{sech}^4\left(\frac{1}{2}(2m+1)\alpha\right)$$
$$= \beta^2 \sum_{m=0}^{\infty} \operatorname{sech}^2\left(\frac{1}{2}(2m+1)\beta\right) - \frac{5}{4}\beta^2 \sum_{m=0}^{\infty} \operatorname{sech}^4\left(\frac{1}{2}(2m+1)\beta\right),$$

which is able to be compared with (1.1).

Corollary 3.3. *For any integer $N \geq 1$,*

$$\sum_{j=0}^{2N-2}{}' (-1)^j E(4N-3, 2N-2-j) \sum_{m=0}^{\infty} \frac{\cosh((2m+1)j\pi)}{\cosh^{4N-2}((2m+1)\pi/2)}$$
$$= \begin{cases} \dfrac{1}{4\pi}, & N = 1, \\ 0, & N > 1, \end{cases}$$

*where $'$ means that the term with $j = 0$ is multiplied by $\frac{1}{2}$.*

*Proof.* Let $B = 0$, $c = 1$ and $\alpha = \beta = \pi$ in Theorem 3.1. Apply

$$\cosh((2m+1)j(\pi - i\pi)) = (-1)^j \cosh((2m+1)j\pi),$$
$$\sinh\left(\frac{1}{2}(2m+1)(\pi - i\pi)\right) = (-1)^{m+1} i \cosh\left(\frac{1}{2}(2m+1)\pi\right).$$

Replace $N$ by $2N-1$ and the desired result follows. $\square$

Corollary 3.3 gives a generalization of (1.2). If $N = 1$ in Corollary 3.3, then we have (1.2).

Corollary 3.4. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any integer $c \geq 1$,*

$$\alpha \sum_{m=0}^{\infty} \operatorname{csch}^2\left(\frac{1}{2c}(2m+1)(\alpha - i\pi)\right) + \beta \sum_{m=0}^{\infty} \operatorname{csch}^2\left(\frac{1}{2c}(2m+1)(\beta + i\pi)\right) = c.$$

*Proof.* Let $B = 0,\ N = 1$ in Theorem 3.1. $\square$

Corollary 3.4 shows a generalized type of formula (1.1). If $c = 1$ in Corollary 3.4, then we have (1.1). Actually (1.2) simply comes from (1.1) when $\alpha = \beta = \pi$. By the way, generalized formulae of them, i.e., Corollary 3.3 and Corollary 3.4 look like quite different formulae.

Now we find another form of infinite series identity. Let $\zeta(s)$ be the Riemann zeta function. Let $\{x\}^* = 1 - \{x\}$. For $k,\ u \in \mathbb{Z}$ and $t,\ v \in \mathbb{C}$, Let

$$g_k(t,u,v) = \begin{cases} (-1)^{[k/2]+u} \sinh(t(u+\frac{1}{2})v), & k \text{ odd}, \\ (-1)^{[k/2]+u} \cosh(tuv), & k \text{ even}. \end{cases}$$

THEOREM 3.5. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any integers $B \geq 0$ and $N \geq 1$,*

$$(-1)^B \alpha^N \sum_{k=0}^{B} \binom{B}{k} \frac{(2\alpha/c)^k}{(2N+k-1)!} {\sum_{j=0}^{N+[k/2]-1}}' E(2N+k-1, N+[k/2]-j-1)$$
$$\times \sum_{m=1}^{\infty} \frac{g_k(2m, j, (\alpha - i\pi)/c)}{m^{-k} \cosh^{2N+k}(m(\alpha - i\pi)/c)}$$
$$= (-\beta)^N \sum_{k=0}^{B} \binom{B}{k} \frac{(2\beta/c)^k}{(2N+k-1)!} {\sum_{j=0}^{N+[k/2]-1}}' E(2N+k-1, N+[k/2]-j-1)$$
$$\times \sum_{m=0}^{\infty} \frac{g_k(2(m+\{c/2\}^*), j, (\beta + i\pi)/c)}{(m+\{c/2\}^*)^{-k} \cosh^{2N+k}((m+\{c/2\}^*)(\beta + i\pi)/c)}$$
$$+ (-1)^B (2^{-1} - 2^{2N-1})(\lambda(c/2)(-1)^B \alpha^{-N} - (-1)^N \beta^{-N}) \zeta(2N) + \delta_N(B, c),$$

*where $'$ means that if $k$ is even, then the term with $j = 0$ is multiplied by $\frac{1}{2}$.*

*Proof.* Let $s_1 = A \geq 1$, $s_2 = -B \leq 0$ and $s = 2N \geq 2$ in Theorem 2.1. Here $A, B$ and $N$ are integers. Put $(r_1, r_2) = (0, \frac{1}{2})$, $(h_1, h_2) = (0, 0)$ and $c\tau - c + 1 = \frac{\pi}{\alpha} i$. Note that

$$V\tau = \frac{1}{c} + i\frac{\alpha}{c\pi} \text{ and } \frac{1}{\Gamma(s_2)} = \frac{1}{\Gamma(-B) = 0}.$$

The direct calculations show that

$$e^{2\pi i((mV\tau + \frac{1}{2})n)} = e^{2\pi i mn(1 + i\frac{\alpha}{\pi})/c + i\pi n} = (-1)^n e^{-2mn(\alpha - i\pi)/c}$$

and

$$U(-B; 2N; 4\pi mn \mathrm{Im}(V\tau)) = (-1)^B (A-1)! \sum_{k=0}^{B} \binom{B}{k} \frac{(-4mn\alpha/c)^k}{(2N+k-1)!}.$$

Thus

$$\mathcal{A}(V\tau, A, B; r, h)$$
$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{e((mV\tau + 1/2)n)}{n^{1-2N}} U(-B; 2N; 4\pi mn \mathrm{Im} V(\tau))$$
$$= (-1)^B (A-1)! \sum_{k=0}^{B} \binom{B}{k} \frac{(-4\alpha/c)^k}{(2N+k-1)!} \sum_{m=1}^{\infty} m^k \sum_{n=1}^{\infty} \frac{(-1)^n e^{-2mn(\alpha - i\pi)/c}}{n^{1-2N-k}}$$
$$= (-1)^B (A-1)! \sum_{k=0}^{B} \binom{B}{k} \frac{(-4\alpha/c)^k}{(2N+k-1)!} \sum_{m=1}^{\infty} \frac{\mathrm{Li}_{-(2N+k-1)}\left(-e^{-2m(\alpha - i\pi)/c}\right)}{m^{-k}}.$$

Put $w = m(\alpha - i\pi)\frac{1}{c}$. Using $E(k,j) = E(k, k-j-1)$ and reversing the summation order over $j$, we find that

$$\begin{aligned}
&\mathrm{Li}_{-(2N+k-1)}(-e^{-2w}) \\
&= \frac{1}{(1+e^{-2w})^{2N+k}} \sum_{j=0}^{2N+k-2} E(2N+k-1, j)(-e^{-2w})^{2N+k-1-j} \\
&= \frac{e^{w(2N+k)}}{(e^w + e^{-w})^{2N+k}} \sum_{j=0}^{2N+k-2} (-1)^{k+j+1} E(2N+k-1, j) e^{w(-4N-2k+2+2j)} \\
&= \frac{(-1)^k 2^{-2N-k}}{\cosh^{2N+k}(w)} \sum_{j=0}^{2N+k-2} (-1)^{j+1} E(2N+k-1, j) e^{-2w(N-k/2-j-1)} \\
&= \frac{(-1)^{N+k} 2^{1-2N-k}}{\cosh^{2N+k}(w)} {\sum_{j=0}^{N+[k/2]-1}}' E(2N+k-1, N+[k/2]-1-j) \\
&\qquad\qquad \times g_k(2m, j, (\alpha - i\pi)/c),
\end{aligned}$$

where $'$ means that if $k$ is even, then the term with $j = 0$ is multiplied by $\frac{1}{2}$. Hence we obtain

$$\begin{aligned}
\mathcal{A}(V\tau, A, -B; r, h) = (-1)^{B+N} \frac{(A-1)!}{2^{2N-1}} &\sum_{k=0}^{B} \binom{B}{k} \frac{(2\alpha/c)^k}{(2N+k-1)!} \\
&\times {\sum_{j=0}^{N+[k/2]-1}}' E(2N+k-1, N+[k/2]-1-j) \\
&\times \sum_{m=1}^{\infty} \frac{m^k g_k(2m, j, (\alpha - i\pi)/c)}{\cosh^{2N+k}(m(\alpha - i\pi)/c)}.
\end{aligned}$$

Note that $\mathcal{A}(V\tau, A, -B; -r, -h) = \mathcal{A}(V\tau, A, -B; r, h)$ for $(r_1, r_2) = (0, \frac{1}{2})$ and $(h_1, h_2) = (0, 0)$. Then

$$\mathcal{H}(V\tau, A, -B; r, h) = 2\mathcal{A}(V\tau, A, -B; r, h).$$

Recalling $\frac{1}{\Gamma(s_2)} = \frac{1}{\Gamma(-B)} = 0$, we find that

$$\begin{aligned}
\mathbf{H}(V\tau, V\bar{\tau}, A, -B; r, h) &= \frac{1}{(A-1)!} \mathcal{H}(V\tau, A, -B; r, h) \\
&= \frac{2}{(A-1)!} \mathcal{A}(V\tau, A, -B; r, h) \\
&= \frac{(-1)^{B+N}}{2^{2N-2}} \sum_{k=0}^{B} \binom{B}{k} \frac{(2\alpha/c)^k}{(2N+k-1)!}
\end{aligned}$$

$$\times \sideset{}{'}\sum_{j=0}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-1-j)$$
$$\times \sum_{m=1}^{\infty} \frac{m^k g_k(2m, j, (\alpha - i\pi)/c)}{\cosh^{2N+k}(m(\alpha - i\pi)/c)}.$$

Next, we examine $\mathbf{H}(\tau, \bar{\tau}, A, -B; R, H)$. Note that

$$(R_1, R_2) = \left(\frac{c}{2}, \frac{1-c}{2}\right), \ (H_1, H_2) = (0, 0), \ \tau = 1 - \frac{1}{c} + i\frac{\pi}{c\alpha}.$$

Replacing $m$ by $m + [-\frac{c}{2}] + 1$ and using $\frac{c}{2} + [-\frac{c}{2}] + 1 = \{\frac{c}{2}\}^*$, we have

$$\mathcal{A}(\tau, A, -B; R, H) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e(((m + \{c/2\}^*)\tau + (1-c)/2)n)}{n^{1-2N}}$$
$$\times U\left(-B; 2N; 4\left(m + \left\{\frac{c}{2}\right\}^*\right)\frac{n\beta}{c}\right).$$

Observe that

$$e(((m + \{c/2\}^*)\tau + (1-c)/2)n) = (-1)^n e^{-2(m+\{c/2\}^*)n(\beta+i\pi)/c}$$

and

$$U\left(-B; 2N; 4\left(m + \left\{\frac{c}{2}\right\}^*\right)\frac{n\beta}{c}\right)$$
$$= (-1)^B (A-1)! \sum_{k=0}^{B} \binom{B}{k} \frac{(-4(m + \{c/2\}^*)n\beta/c)^k}{(2N+k-1)!}.$$

Thus, applying the same method to compute $\mathcal{A}(\tau, A, -B; r, h)$, we find

$$\mathcal{A}(\tau, A, -B; R, H)$$
$$= (-1)^B (A-1)! \sum_{k=0}^{B} \binom{B}{k} \frac{(-4\beta/c)^k}{(2N+k-1)!} \sum_{m=0}^{\infty} \left(m + \left\{\frac{c}{2}\right\}^*\right)^k$$
$$\times \mathrm{Li}_{-(2N+k-1)}(-e^{-2(m+\{c/2\}^*)(\beta+i\pi)/c})$$
$$= (-1)^{B+N} \frac{(A-1)!}{2^{2N-1}} \sum_{k=0}^{B} \binom{B}{k} \frac{(2\beta/c)^k}{(2N+k-1)!}$$
$$\times \sideset{}{'}\sum_{j=1}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-1-j)$$
$$\times \sum_{m=0}^{\infty} \frac{(m + \{c/2\}^*)^k g_k(2(m + \{c/2\}^*), j, (\beta + i\pi)/c)}{\cosh^{2N+k}(2(m + \{c/2\}^*)(\beta + i\pi)/c)}.$$

Employing $\mathcal{A}(\tau, A, -B; -R, -H) = \mathcal{A}(\tau, A, -B; R, H)$ and $\frac{1}{\Gamma(-B)} = 0$, we finally have

$$\begin{aligned}\mathbf{H}(\tau,\bar{\tau};A,-B;R,H) &= \frac{2}{(A-1)!}\mathcal{A}(\tau,A,-B;R,H)\\ &= \frac{(-1)^{B+N}}{2^{2N-2}}\sum_{k=0}^{B}\binom{B}{k}\frac{(2\beta/c)^k}{(2N+k-1)!}\\ &\times \sum_{j=1}^{N+[k/2]-1}{}' E(2N+k-1, N+[k/2]-1-j)\\ &\times \sum_{m=0}^{\infty}\frac{(m+\{c/2\}^*)^k g_k(2(m+\{c/2\}^*), j, (\beta+i\pi)/c)}{\cosh^{2N+k}(2(m+\{c/2\}^*)(\beta+i\pi)/c)}.\end{aligned}$$

The term with $\lambda(R_1) = \lambda(\frac{c}{2})$ is valid only when $c$ is even. For $c$ even,

$$\begin{aligned}\Psi(-H_2, R_2, 2N) &= 2\psi(0,(c-1)/2,2N)\\ &= 2\sum_{n+(c-1)/2>0}\frac{1}{(n+(c-1)/2)^{2N}}\\ &= 2\sum_{n=0}^{\infty}\frac{1}{(n+1/2)^{2N}} = 2(2^{2N}-1)\zeta(2N).\end{aligned}$$

Similarly we have

$$\Psi(h_2, r_2, 2N) = \Psi(0, 1/2, 2N) = 2(2^{2N}-1)\zeta(2N).$$

The terms with $\lambda(H_2)$ and $\lambda(h_2)$ are nullified by using $\frac{1}{\Gamma(-B)} = 0$. Lastly, it is easy to see that $L(\tau, \bar{\tau}, A, -B; R, H)$ has the same evaluation in the proof of Theorem 3.1. Using all the above results, we arrive at the desired result. □

Let $c = 1$ in Theorem 3.5. Short calculations show that

$$\begin{aligned}&\sinh(m(2j+1)(\alpha-i\pi)) = (-1)^m \sinh(m(2j+1)\alpha),\\ &\cosh(2mj(\alpha-i\pi)) = \cosh(2mj\alpha),\\ &\cosh^{2N+k}(m(\alpha-i\pi)) = (-1)^{mk}\cosh^{2N+k}(m\alpha)\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\cosh((2m+1)j(\beta+i\pi)) = (-1)^j\cosh((2m+1)j\beta),\\ &\sinh((2m+1)(j+1/2)(\beta+i\pi)) = (-1)^{m+j} i\cosh((2m+1)(j+1/2)\beta),\\ &\cosh^{2N+k}((m+1/2)(\beta+i\pi)) = (-1)^{N+mk} i^k \sinh^{2N+k}((m+1/2)\beta).\end{aligned} \tag{3.3}$$

Now we obtain identities with only real values.

COROLLARY 3.6. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any integers $B \geq 0$ and $N \geq 1$ with $B \equiv N \pmod 2$,*

$$\alpha^N \sum_{k=0}^{B} \binom{B}{k} \frac{(2\alpha)^k}{(2N+k-1)!} \sideset{}{'}\sum_{j=0}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-j-1) \times \sum_{m=1}^{\infty} \frac{g_k(2m,j,\alpha)}{m^{-k}\cosh^{2N+k}(m\alpha)}$$
$$= \beta^N \sum_{k=0}^{B} \binom{B}{k} \frac{\beta^k}{(2N+k-1)!} \sideset{}{'}\sum_{j=0}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-j-1) \times \sum_{m=0}^{\infty} \frac{\cosh((2m+1)(j+\mu_k)\beta)}{(2m+1)^{-k}\sinh^{2N+k}((2m+1)\beta/2)}$$
$$+(2^{2N-1}-2^{-1})(-\beta)^{-N}\zeta(2N).$$

*Proof.* Put $c = 1$ in Theorem 3.5. Employing (3.2), (3.3), we have

$$\frac{g_k(2m,j,\alpha - i\pi)}{\cosh^{2N+k}(m(\alpha - i\pi))} = \frac{g_k(2m,j,\alpha)}{\cosh^{2N+k}(m\alpha)}$$

and

$$\frac{g_k(2(m+\{1/2\}^*), j, \beta + i\pi)}{\cosh^{2N+k}((m+\{1/2\}^*)(\beta + i\pi))} = (-1)^N \frac{\cosh((2m+1)(j+\mu_k)\beta)}{\sinh^{2N+k}((m+1/2)\beta)}.$$

Use the above calculations to obtain the desired result. □

COROLLARY 3.7. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any integer $N \geq 1$,*

$$\alpha^N \sideset{}{'}\sum_{j=0}^{N-1} (-1)^j E(2N-1, N-1-j) \sum_{m=1}^{\infty} \frac{\cosh(2mj\alpha)}{\cosh^{2N}(m\alpha)}$$
$$= \beta^N \sideset{}{'}\sum_{j=0}^{N-1} E(2N-1, N-1-j) \sum_{m=0}^{\infty} \frac{\cosh((2m+1)j\beta)}{\sinh^{2N}((2m+1)\beta/2)}$$
$$+(-\beta)^{-N}(2N-1)!(2^{2N-1}-2^{-1})\zeta(2N) + \epsilon_N,$$

*where $'$ means that the term with $j = 0$ is multiplied by $\frac{1}{2}$ and*

$$\epsilon_N = \begin{cases} \frac{1}{2}, & N = 1, \\ 0, & N \geq 2. \end{cases}$$

*Proof.* Put $B = 0$ and $c = 1$ in Theorem 3.5 and apply (3.2), (3.3). □

Corollary 3.7 is a generalization of (1.3). If $N = 1$ in Corollary 3.7, then we find (1.3).

Corollary 3.8. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any integer $B \geq 0$,*

$$
\begin{aligned}
&(-1)^B \alpha \sum_{k=1}^{B} \binom{B}{k} \frac{(2\alpha)^k}{(k+1)!} {\sum_{j=0}^{[k/2]}}' E(k+1, [k/2]-j) \sum_{m=1}^{\infty} \frac{g_k(2m, j, \alpha)}{m^{-k}\cosh^{k+2}(m\alpha)} \\
&+(-1)^B \frac{\alpha}{2} \sum_{m=1}^{\infty} \operatorname{sech}^2(m\alpha) + (-1)^B \frac{\alpha}{4} \\
&= \beta \sum_{k=1}^{B} \binom{B}{k} \frac{\beta^k}{(k+1)!} {\sum_{j=0}^{[k/2]}}' E(k+1, [k/2]-j) \\
&\qquad\qquad \times \sum_{m=1}^{\infty} \frac{\cosh((2m+1)(j+\mu_k)\beta)}{(2m+1)^{-k}\sinh^{k+2}((2m+1)\beta/2)} \\
&\quad + \frac{\beta}{2} \sum_{m=0}^{\infty} \operatorname{csch}^2\left(\frac{1}{2}(2m+1)\beta\right) + \frac{1+(-1)^B}{4(B+1)},
\end{aligned}
$$

*where $'$ means that if $k$ is even, then the term with $j = 0$ is multiplied by $\frac{1}{2}$.*

*Proof.* Put $N = 1$ and $c = 1$ in Theorem 3.5 and use (3.2), (3.3). □

We also see that Corollary 3.8 includes (1.3). We may obtain more new types of infinite series identities using Theorem 2.1 as B. C. Berndt has done in [2, 3].

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